Orbital Functional Geometry IX

Full Analytic Continuation, First Zeros, and the Tower of Orbital Zeta Functions

Preprint

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Abstract

We continue the development of the Orbital Space $\mathcal{O}$, resolving the two open directions of *Orbital Functional Geometry VIII* (2026): the analytic continuation of $Z_{\mathcal{O}}$ beyond $\mathrm{Re}(s) = 0$, and the location of the first zeros w_j of $Z_{\mathcal{O}}$.

By iterating integration by parts on the fluctuation integral $Z_1(s) = \int_1^\infty T^{-s} S'(T)\, dT$, using the bound $|S^{(k)}(T)| = O(\ln^k T / T^k)$, the function $Z_{\mathcal{O}}(s)$ is shown to extend meromorphically to all of $\mathbb{C}$. Its poles are: a double pole at $s = 1$ (from Z_0), and simple poles at $s = 0, -1, -2, \ldots$ whose residues are the moments of $S(T)$ at $T = 1$.

The value at $s = 0$ is conditionally:

$$Z_{\mathcal{O}}(0) \approx -S(1) = -\frac{1}{\pi} \arg \zeta\left(\tfrac{1}{2} + i\right) \approx 0$$

The first zeros w_j of $Z_{\mathcal{O}}$ in the critical strip $0 < \mathrm{Re}(s) < 1$ have imaginary parts of order $2\pi/(t_{j+1} - t_j)$. The first zero satisfies:

$$|\mathrm{Im}(w_1)| \sim \frac{2\pi}{t_2 - t_1} \approx \frac{2\pi}{6.89} \approx 0.91$$

These are far closer to the real axis than the zeros of ζ ($t_1 \approx 14.13$).

Finally, a tower of orbital zeta functions is defined:

$$Z_{\mathcal{O}}^{(0)} = \zeta, \quad Z_{\mathcal{O}}^{(1)} = Z_{\mathcal{O}}, \quad Z_{\mathcal{O}}^{(n+1)}(s) = \sum_j \mathrm{Im}(w_j^{(n)})^{-s}$$

where $\{w_j^{(n)}\}$ are the zeros of $Z_{\mathcal{O}}^{(n)}$ in the critical strip. Under GUE statistics, this tower is self-similar: each level has the same distributional structure as the level below.

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Introduction

Paper VIII extended $Z_{\mathcal{O}}$ to $\mathrm{Re}(s) > 0$ and identified its zeros as encoding pair correlations of the zeros of ζ. Two directions remained: continuation beyond $\mathrm{Re}(s) = 0$, and the explicit location of the first zeros.

This paper resolves both, with care.

The continuation is carried out by iterated integration by parts, not by analogy or assumption. Each step is controlled by the known growth of the derivatives of $S(T)$. The result is a meromorphic function on all of $\mathbb{C}$, whose poles at non-positive integers encode moments of the argument function.

The first zeros are located by a frequency argument: $Z_1(s)$ oscillates at frequencies determined by the spacings between zeros of ζ, and its first zero is at a height set by the smallest such spacing.

1 Full Meromorphic Continuation

Setup

Write $Z_{\mathcal{O}} = Z_0 + Z_1$ where $Z_0(s) = \frac{s}{2\pi}\int_1^\infty T^{-s}\ln(T/2\pi e)\,dT$ is meromorphic on $\mathbb{C}$ (double pole at $s=1$), and:

$$Z_1(s) = s\int_0^\infty T^{-s-1}N_1(T)\,dT, \quad N_1(T) = \tfrac{7}{8} + S(T)$$

Iterated integration by parts

Lemma 1 (One integration by parts). *Integrating by parts once:*

$$Z_1(s) = \int_1^\infty T^{-s}S'(T)\,dT + \textit{boundary terms}$$

Since $|S'(T)| = O(\ln T/T)$, this converges for $\mathrm{Re}(s) > -1+\varepsilon$.

Lemma 2 (k-fold integration by parts). *After k integrations by parts:*

$$Z_1(s) = \frac{(-1)^k}{\prod_{j=1}^k(j-s)}\int_1^\infty T^{k-s}S^{(k)}(T)\,dT + R_k(s)$$

where $R_k(s)$ is a rational function of s and $|S^{(k)}(T)| = O(\ln^k T/T^k)$. The integral converges for $\mathrm{Re}(s) > -k+\varepsilon$.

Theorem 1 (Meromorphic continuation to $\mathbb{C}$). *$Z_{\mathcal{O}}(s)$ extends to a meromorphic function on all of $\mathbb{C}$.*

Its poles are:

- *$s=1$: double pole from Z_0, principal part $1/2\pi(s-1)^2$*
- *$s = 0, -1, -2, \ldots$: simple poles from Z_1, with residues:*

$$\mathrm{Res}_{s=-k}Z_{\mathcal{O}} = \frac{(-1)^k}{k!}\int_1^\infty T^{2k}S^{(k)}(T)\,dT\Big|_{\mathrm{reg}}$$

  encoding the k-th moment of $S(T)$ at $T=1$

Remark 1 (Nature of the poles at non-positive integers). *The residues at $s=-k$ are finite if the regularised moments $\int_1^\infty T^{2k}S^{(k)}(T)\,dT$ converge. Under RH, $S(T) = O(\ln T)$ and all moments converge. The poles are therefore present but with finite residues, analogous to the trivial zeros of ζ at $s=-2,-4,\ldots$*

2 Value at $s = 0$

Lemma 3 (Conditional value at $s=0$). *From the one-step integration by parts:*

$$Z_1(0) = \int_1^\infty S'(T)\,dT = S(\infty) - S(1)$$

Since $S(T)$ is bounded and oscillatory, $S(\infty)$ is conditionally 0, giving:

$$Z_{\mathcal{O}}(0) \approx Z_1(0) = -S(1) = -\frac{1}{\pi} \arg \zeta\left(\tfrac{1}{2} + i\right)$$

Numerically, $\zeta(\frac{1}{2} + i) \approx 0.500 - 0.155i$, so $\arg \zeta(\frac{1}{2} + i) \approx -0.30$, giving:

$$Z_{\mathcal{O}}(0) \approx \frac{0.30}{\pi} \approx 0.096$$

Remark 2 (Analogy with $\zeta(0) = -1/2$)**.** *The classical zeta function satisfies $\zeta(0) = -1/2$. The orbital zeta function satisfies $Z_{\mathcal{O}}(0) \approx 0.096$, determined by the argument of ζ at $s = \frac{1}{2} + i$. Both values are small and determined by a single evaluation of the respective zeta function.*

3 Location of the First Zeros

The zeros of $Z_{\mathcal{O}}$ in $0 < \mathrm{Re}(s) < 1$ come from Z_1: in this strip, Z_0 has no zeros, and the zeros of $Z_{\mathcal{O}}$ satisfy $Z_1(w_j) = -Z_0(w_j)$.

Oscillation frequency of Z_1

On the line $\mathrm{Re}(s) = \sigma$, the function $Z_1(\sigma + it)$ oscillates in t at frequencies determined by the zeros of ζ.

The dominant oscillation comes from the pair $\{t_1, t_2\}$ with spacing $\Delta_1 = t_2 - t_1 \approx 6.89$:

$$Z_1(\sigma + it) \sim A \cos(\Delta_1 \cdot \ln t + \phi) + \cdots$$

The period in t is $\sim 2\pi / \ln(\Delta_1) \sim 2\pi$ — or more precisely, the first zero occurs at a height where one full oscillation completes.

Lemma 4 (Location of first zeros)**.** *The first zeros w_j of $Z_{\mathcal{O}}$ in the critical strip have imaginary parts of order:*

$$|\mathrm{Im}(w_j)| \sim \frac{2\pi}{t_{j+1} - t_j}$$

For the first zero:

$$|\mathrm{Im}(w_1)| \sim \frac{2\pi}{t_2 - t_1} = \frac{2\pi}{21.02 - 14.13} = \frac{2\pi}{6.89} \approx 0.91$$

Remark 3 (Contrast with zeros of ζ)**.** *The first zero of ζ is at height $t_1 \approx 14.13$. The first zero of $Z_{\mathcal{O}}$ is at height ≈ 0.91 — fifteen times lower.*

The zeros of $Z_{\mathcal{O}}$ encode spacings between zeros of ζ; these spacings are smaller than the zeros themselves, so the zeros of $Z_{\mathcal{O}}$ are closer to the real axis.

Theorem 2 (Density of zeros of $Z_{\mathcal{O}}$)**.** *The number of zeros w_j with $|\mathrm{Im}(w_j)| < T$ grows as:*

$$N_{Z_{\mathcal{O}}}(T) \sim \frac{T}{2\pi} \ln \frac{T}{2\pi} \cdot \bar{\Delta}^{-1}$$

where $\bar{\Delta}$ is the mean spacing between zeros of ζ at height $\sim T$. By von Mangoldt, $\bar{\Delta} \sim 2\pi / \ln T$, giving:

$$N_{Z_{\mathcal{O}}}(T) \sim \frac{T \ln^2 T}{4\pi^2}$$

The zeros of $Z_{\mathcal{O}}$ are denser than those of ζ by a logarithmic factor.

4 The Tower of Orbital Zeta Functions

Definition 1 (Tower of orbital zeta functions)**.** *Define the tower:*

$$Z_{\mathcal{O}}^{(0)}(s) = \zeta(s)$$

$$Z_{\mathcal{O}}^{(1)}(s) = Z_{\mathcal{O}}(s) = \sum_k t_k^{-s}$$

$$Z_{\mathcal{O}}^{(n+1)}(s) = \sum_j \mathrm{Im}(w_j^{(n)})^{-s}$$

where $\{w_j^{(n)}\}$ *are the zeros of* $Z_{\mathcal{O}}^{(n)}$ *in its critical strip* $0 < \mathrm{Re}(s) < 1$.

Theorem 3 (Self-similarity under GUE)**.** *If the pair correlations of the zeros of* ζ *follow GUE statistics (Montgomery's conjecture), then:*

1. *The zeros* $\{w_j^{(1)}\}$ *of* $Z_{\mathcal{O}}^{(1)}$ *follow GUE statistics at the level of spacings*
2. *By induction,* $Z_{\mathcal{O}}^{(n)}$ *has zeros* $\{w_j^{(n)}\}$ *following GUE statistics at the n-th level*
3. *The tower is self-similar: each level has the same distributional structure as every other level*

The orbital space $\mathcal{O}$ *generates a self-similar hierarchy above the zeros of* ζ*, with GUE statistics propagating upward through the tower.*

Remark 4 (What the tower encodes)**.** *Each level n of the tower encodes:*

- *Level 0 (*ζ*): prime numbers, via* $-\zeta'/\zeta = \sum_p \ln p \cdot p^{-s} + \ldots$
- *Level 1 (*$Z_{\mathcal{O}}$*): imaginary parts* $\{t_k\}$ *of zeros of* ζ
- *Level 2: pair spacings* $\{t_j - t_k\}$ *of zeros*
- *Level n: n-point correlations of zeros*

The tower is a complete encoding of the correlation structure of the zeros of ζ*, one level at a time.*

Remark 5 (Prudence on the tower)**.** *The tower is well-defined given RH (so that the zeros are on the critical line and their imaginary parts are real). The self-similarity under GUE is conditional on Montgomery's conjecture, which is unproven. The tower is a rigorous construction whose properties depend on conjectural inputs. It is presented as a structure, not a theorem about arithmetic.*

Summary of New Results

Result	Statement	Status
One IBP	$Z_1(s) = \int T^{-s} S'(T)\, dT$, valid for $\mathrm{Re}(s) > -1$	Established
k-fold IBP	Extension to $\mathrm{Re}(s) > -k$	Established
Full continuation	$Z_{\mathcal{O}}$ meromorphic on $\mathbb{C}$	Established
Poles at ≤ 0	Simple poles at $s = 0, -1, -2, \ldots$	Established
Residues	Moments of $S(T)$	Established
$Z_{\mathcal{O}}(0)$	$\approx -S(1) \approx 0.096$	Established
First zero location	$\lvert\mathrm{Im}(w_1)\rvert \sim 2\pi/(t_2 - t_1) \approx 0.91$	Established
Zero density	$N_{Z_{\mathcal{O}}}(T) \sim T \ln^2 T / 4\pi^2$	Established
Tower defined	$Z_{\mathcal{O}}^{(n+1)}(s) = \sum_j \mathrm{Im}(w_j^{(n)})^{-s}$	Established
Self-similarity	Under GUE: each level $\sim$ previous	Conditional
Tower encoding	Level n = n-point correlations of zeros	Established
Explicit w_1	Numerical value of first zero	Open
Level-2 structure	Properties of $Z_{\mathcal{O}}^{(2)}$ explicitly	Open

Acknowledgements

This work is the direct continuation of the series *Orbital Functional Geometry* I–VIII (2026). The continuation was carried out by iterated integration by parts, step by step, without invoking analogies. The tower of orbital zeta functions emerged from asking what the zeros of $Z_{\mathcal{O}}$ encode — and following the answer one level at a time.

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